A GENERALIZED SIMILARITY METHOD WITH A UNIVERSAL EQUATION IN DIFFERENTIAL FORM IN THE THEORY OF A NONSTATIONARY BOUNDARY LAYER

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The results of the integration of a "segment" of a universal equation in purely differential form for a nonstationary laminar boundary layer are presented and analyzed.

A universal equation for a nonstationary laminar boundary layer in an incompressible liquid was obtained in [1] without the use of any integral equations, i.e., in purely differential form. We write this equation and the boundary conditions for the dimensionless stream function φ in the following form:

$$B^2\frac{\partial^3\varphi}{\partial\eta^3}+f_{10}\left[\varphi\frac{\partial^2\varphi}{\partial\eta^2}-\left(\frac{\partial\varphi}{\partial\eta}\right)^2+1\right]+f_{01}\left(1-\frac{\partial\varphi}{\partial\eta}\right)+$$
$$+\frac{\varphi g_{10}+\eta g_{01}}{2}\frac{\partial^2\varphi}{\partial\eta^2}=\sum_{k,l,m,n=0}^{\infty}\left\{\frac{\partial^2\varphi}{\partial f_{kn}\partial\eta}[(k-1)f_{01}f_{kn}+(k+n)g_{01}f_{kn}+f_{k,n+1}]+\right.$$
$$+\frac{\partial^2\varphi}{\partial g_{lm}\partial\eta}[lf_{01}g_{lm}+(l+m-1)g_{01}g_{lm}+g_{l,m+1}]+$$
$$+\left(\frac{\partial\varphi}{\partial\eta}\frac{\partial^2\varphi}{\partial f_{kn}\partial\eta}-\frac{\partial^2\varphi}{\partial\eta^2}\frac{\partial\varphi}{\partial f_{kn}}\right)[(k-1)f_{10}f_{kn}+(k+n)g_{10}f_{kn}+f_{k+1,n}]+$$
$$\left.+\left(\frac{\partial\varphi}{\partial\eta}\frac{\partial^2\varphi}{\partial g_{lm}\partial\eta}-\frac{\partial^2\varphi}{\partial\eta^2}\frac{\partial\varphi}{\partial g_{lm}}\right)[lf_{10}g_{lm}+(l+m-1)g_{10}g_{lm}+g_{l+1,m}]\right\}, \quad (1)$$

$$\varphi=0,\quad \frac{\partial\varphi}{\partial\eta}=0 \ \text{at}\ \ \eta=0;\quad \frac{\partial\varphi}{\partial\eta}\to 1\ \ \text{as}\ \ \eta\to\infty; \quad (2)$$
$$\varphi=\varphi_a(\eta)\ \ \text{at}\ \ f_{kn}=(f_{kn})_a\ \ \text{and}\ \ g_{lm}=(g_{lm})_a,$$

where

$$\varphi(\eta, f_{kn}, g_{lm})=\frac{B\psi(x, y, t)}{U(x, t)h(x, t)};\quad \eta=\frac{By}{h(x, t)}. \quad (3)$$

The series of parameters introduced, which replace the longitudinal coordinate x and the time t, has the form

$$f_{kn}=U^{k-1}\frac{\partial^{k+n}U}{\partial x^k\partial t^n}z^{k+n},\ g_{lm}=U^l\frac{\partial^{l+m}z}{\partial x^l\partial t^m}z^{l+m-1}, \quad (4)$$

where $z=h^2/\nu$ and k, n, l, m = 0, 1, 2, ...

With arbitrary functions U(x, t) and h(x, t) the parameters f_{kn} and g_{lm} are independent variables. A connection between them is established only in the second stage during the solution of a particular problem, when a concrete scale is chosen for the transverse coordinate. In the boundary conditions (2) the function $\varphi_a(\eta)$ is some self-similar solution. If as the latter one chooses the Blasius solution for a steady boundary layer at a plate ($\varphi_a=\varphi_0$), then the second line of the boundary conditions (2) will be

$$\left.\begin{array}{r}\varphi=\varphi_0(\eta)\ \ \text{at}\ \ f_{kn}=0,\quad g_{10}=\beta=\text{const},\quad g_{l0}=0\ (l=2,3,\ldots)\\ g_{lm}=0\ (l=0,1,2,\ldots;\quad m=1,2,3,\ldots)\end{array}\right\}. \quad (5)$$

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TABLE 1. Values of the Coefficients in the Expansions (14)-(16)

Parameters	Functions			Parameters	Functions		
	ζ	H^*	H^{**}		ζ	H^*	H^{**}
1	0,2205	2,5915	0,9998	f_{10}^2	6,8412	81,4026	6,2082
f_{10}	2,3236	—12,3480	—2,6752	f_{01}^2	0,9309	—5,9575	—3,9436
f_{01}	1,2762	—3,5069	—0,0271	g_{10}^{*2}	—0,1417	4,9975	1,9272
g_{10}^*	0,2501	—2,9399	—1,1349	g_{01}^2	—1,4335	33,9696	10,1525
g_{01}	0,7508	—7,3501	—2,4768	f_{20}	—0,6952	6,1282	1,7965
$f_{10}f_{01}$	—4,5236	35,6690	—6,5571	f_{02}	—1,6086	9,2476	1,7298
$f_{10}g_{10}^*$	—2,6369	42,0367	9,1004	f_{11}	—4,4305	30,3642	7,2237
$f_{10}g_{01}$	—10,5547	130,3947	25,0552	g_{20}	—0,2838	3,3350	1,2872
$f_{01}g_{10}^*$	—1,4523	11,9572	0,0972	g_{02}	—2,7367	23,3554	6,9863
$f_{01}g_{01}$	—3,2934	24,5847	—2,5644	g_{11}	—1,7367	16,6760	5,6188
$g_{10}^*g_{01}$	—0,8498	24,9968	8,4151				

The method of series expansion around values of the parameters corresponding to self-similar motion is effective in the integration of a "segment" of the universal equation (1), when a limited number of parameters are retained, for flows in a nearly self-similar boundary layer. In the present work such an expansion was constructed near the solution φ_0 of the Blasius problem. It is known that the characteristic transverse scales (δ^* is the displacement thickness and δ^{**} is the thickness of momentum loss) in this problem can be expressed by the general equation [2]

$$h^0 = \sqrt{\beta \frac{\nu x}{U_\infty}} \tag{6}$$

or

$$z^0 = \frac{(h^0)^2}{\nu} = \beta \frac{x}{U_\infty}, \quad g_{10}^0 = U_\infty (z^0)' = \beta, \tag{7}$$

where the constant takes the values $\beta = 3$ at $h^0 = \delta^*$ and $\beta = 0.4408$ at $h^0 = \delta^{**}$. We note, however, that $g_{10}^0 = \beta = 0.4408$ in accordance with Eq.(6) and in the case when $h^0 = 0.383\delta^*$. The latter circumstance makes it possible to obtain the value of the normalizing constant B independent of the choice of a concrete scale h. In fact, when the conditions (5) are observed Eq. (1) becomes the Blasius equation

$$\frac{d^3\varphi_0}{d\eta^3} + \frac{\beta}{2B^2}\varphi_0\frac{d^2\varphi_0}{d\eta^2} = 0, \tag{8}$$

in which case it is convenient to take

$$B^2 = \beta/2. \tag{9}$$

Setting $\beta = 0.4408$, we obtain B = 0.47. This value of the normalizing constant, which will be used in the integration of Eq. (1), corresponds to two scales, as shown below:

$$h = \delta^{**} \text{ and } h = 0.383\,\delta^*. \tag{10}$$

We note that the value of the normalizing constant could be taken as B = 1. Then numerical coefficients differing from those obtained would appear in both the equalities (10). From this it follows that Eq. (1) is also "universal" relative to the choice of the scale h. This property will be used later in solving a particular problem.

We will seek the solution of (1) in the form of a "segment" of the power series

$$\varphi = \varphi_0(\eta) + \varphi_1(\eta) f_{10} + \varphi_2(\eta) f_{01} + \varphi_3(\eta) g_{10}^* + \varphi_4(\eta) g_{01} + \varphi_5(\eta) f_{10}f_{01} + \cdots + \varphi_{19}(\eta) g_{02} + \varphi_{20}(\eta) g_{11}, \tag{11}$$

where

$$g_{10}^* = g_{10} - g_{10}^0 = g_{10} - \beta. \tag{12}$$

Substituting the polynomial (11) into Eq. (1) and equating the coefficients to equal one-term combinations of parameters, we find for $\varphi_i(\eta)$ a system of ordinary differential equations. As a result of a numerical solution of these equations on a computer, carried out once and for all with the boundary conditions:

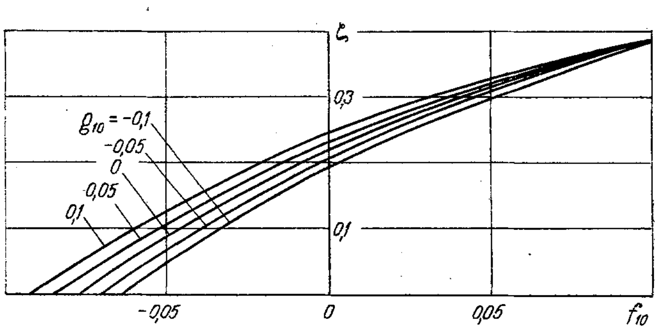

Fig. 1. Dependence of reduced coefficient of friction on the parameters f_{10} and g_{10}; $f_{01} = f_{20} = f_{02} = f_{11} = g_{01} = g_{20} = g_{02} = 0$.

$$\varphi_0 = \frac{d\varphi_0}{d\eta} = 0 \text{ at } \eta = 0, \quad \frac{d\varphi_0}{d\eta} \to 1 \text{ as } \eta \to \infty;$$
$$\varphi_i = \frac{d\varphi_i}{d\eta} = 0 \text{ at } \eta = 0, \quad \frac{d\varphi_i}{d\eta} \to 0 \text{ as } \eta \to \infty \; (i = 1, 2, \ldots), \tag{13}$$

we obtained the values of the coefficients both for the polynomial (11) and for expansions of the following characteristic functions:

$$\zeta = \frac{\tau_w h}{\mu U} = B \left. \frac{\partial^2 \varphi}{\partial \eta^2} \right|_{\eta=0} = \zeta_0 + \zeta_1 f_{10} + \zeta_2 f_{01} + \cdots + \zeta_{20} g_{11}, \tag{14}$$

$$H^* = \frac{\delta^*}{h} = \frac{1}{B} \int_0^\infty \left(1 - \frac{\partial \varphi}{\partial \eta}\right) d\eta = H_0^* + H_1^* f_{10} + H_2^* f_{01} + \cdots + H_{20}^* g_{11}, \tag{15}$$

$$H^{**} = \frac{\delta^{**}}{h} = \frac{1}{B} \int_0^\infty \frac{\partial \varphi}{\partial \eta} \left(1 - \frac{\partial \varphi}{\partial \eta}\right) d\eta = H_0^{**} + H_1^{**} f_{10} + H_2^{**} f_{01} + \cdots + H_{20}^{**} g_{11}. \tag{16}$$

The values of the coefficients to the parameters and of their combinations in the expansions (14)-(16) are given in Table 1.

An analysis of the functions obtained made it possible to determine the character of the effect of the parameters on some properties of a nonstationary boundary layer and, in particular, on the value of the reduced surface friction and the value of the separation abscissa. The dependences of these characteristics on the parameters f_{10}, f_{01}, and g_{01} revealed in [3] were confirmed in the solution under consideration. An original result was obtained in an analysis of the variation of g_{10}^*, which reflects the past history of the flow in the boundary layer. It follows from Fig. 1 that the effect of this parameter shows up mainly in the diffuse region with $f_{10} < 0$. With an increase in the positive value of g_{10}^*, which corresponds to an increase in the parameter $g_{10} = Uz'$, the friction in the boundary layer increases and separation is accordingly protracted. An analysis of the effect of f_{20}, f_{02}, f_{11}, g_{20}, g_{02}, and g_{11} showed a tendency opposite to that just presented.

In solving a concrete problem with a given velocity distribution U(x, t) at the outer limit of the boundary layer it is necessary to determine the quantity z(x, t) after which the values of the parameters become known and, consequently, the velocity profiles and the surface friction in the boundary layer become known as functions of the longitudinal coordinate and time in accordance with the solution of the universal equation. The integral momentum equation, reduced to parametric form with an arbitrary scale h, is an identity and therefore cannot be used to determine the quantity z. A concrete assignment of the scale in one of the forms (10) leads to the fact that some of the terms drop out of the integral momentum equation and it becomes the equation from which z can be determined. Thus, with $h = \delta^{**}$ we have

$$\frac{\dot{z} H^*}{2} + z \frac{\partial H^*}{\partial t} + \frac{\dot{U} z}{U} H^* + \frac{U z'}{2} + U' z (2 + H^*) - \zeta = 0, \tag{17}$$

while with $h = 0.383\delta^*$ we have

$$\frac{1}{0.383} \left(\frac{\dot{z}}{2} + \frac{\dot{U} z}{U}\right) + \frac{U z'}{2} H^{**} + U z \frac{\partial H^{**}}{\partial x} + U' z \left(2H^{**} + \frac{1}{0.383}\right) - \zeta = 0. \tag{18}$$

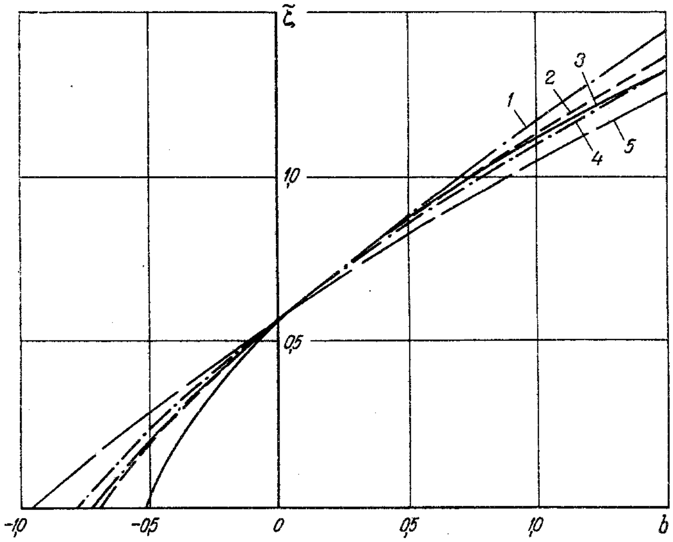

Fig. 2. Dependence of reduced coefficient of friction $\tilde{\zeta}$ on exponent b. (All quantities are dimensionless.)

The functionals ζ, H^*, and H^{**} entering into the equations were determined from Eqs. (14)-(16) while the derivatives $\partial H^*/\partial t$ and $\partial H^{**}/\partial x$ were calculated with allowance for the recurrent relations of [1]. Thus, to determine z(x, t) we can use either of the equations (17) or (18) obtained, which are differential equations in partial derivatives, with their order being determined by the numbers of parameters g_{lm} contained in the expansions of H^*, H^{**}, and ζ for the adopted approximation. Being confined to the parameters g_{10} and g_{01}, we obtain first-order differential equations, which we will later integrate. We note that the discarded terms, which contain higher-order derivatives, require special investigation, generally speaking.

Besides the method presented above, which is standard, one can use the expansions (15) and (16) of the characteristic functions for the determination of z(x, t), equating the polynomials obtained in the approximation under consideration to certain constants in accordance with the chosen scales. For example, being confined to five terms of the expansion, with $h = \delta^{**}$ we can write

$$H^{**} = 1 = H_0^{**} + H_1^{**} U' z + H_2^{**} \frac{\dot{U} z}{U} + H_3^{**} (U z' - \beta) + H_4^{**} \dot{z}, \tag{19}$$

while with $h = 0.383\delta^*$

$$H^* = \frac{1}{0.383} = 2.61 = H_0^* + H_1^* U' z + H_2^* \frac{\dot{U} z}{U} + H_3^* (U z' - \beta) + H_4^* \dot{z}. \tag{20}$$

As an example of the calculation, let us consider the problem of the growth of the boundary at a plate, infinite in both directions, moving in its own plane. Let the velocity at the boundary-layer limit vary with time by the power law

$$U = a t^b, \tag{21}$$

where a and b are constants ($a > 0$). We will seek the quantity z in the form

$$z = \lambda(b) t. \tag{22}$$

Substituting the expressions (21) and (22) into one of the equations (17)-(20), we obtain an algebraic equation for the calculation of $\lambda(b)$. Using (21), we will have $g_{01} = \lambda(b)$, $f_{01} = b\lambda(b)$, and $f_{02} = b(b-1)\lambda(b)$; the remaining parameters are equal to zero. The results of the calculation in the form of the reduced friction $\tilde{\zeta}$ as a function of the exponent b are presented in Fig. 2. Here $\tilde{\zeta} = (\tau_w/\mu U)\sqrt{\nu t} = \zeta/\sqrt{\lambda(b)}$. The exact solution [4] is shown by the solid curve 3. Using the arbitrariness in the choice of the transverse scale, as well as the equations for finding z(x, t), we were able to find an approximation solution by four methods. Curves 1 and 5 were obtained with $h = \delta^{**}$ while curves 2 and 4 were obtained with $h = 0.383\delta^*$. Curves 2 and 5 were found using the momentum equations (17) and (18), respectively, to find z, while curves 1 and 4 were found using (19) and (20), respectively, for this. It should be noted that with $b > 0$ the solution was obtained in a quadratic approximation while with $b < 0$ it was obtained in a linear approximation. The latter is explained by the fact that in this case the quadratic approximation led to the appearance of complex solutions of the quadratic equation relative to $\lambda(b)$, since a negative difference of two quantities close in absolute value was formed under the square root. It would be possible to avoid this by allowing for the cubic terms in the expansion of the unknown functions. It is seen from an examination of the graphs that the solution using the displacement thickness as the scale gives the best approximation to the exact solution.

The method presented in the article cannot be used to calculate a nonsteady boundary layer at a body with a sharp leading edge, since the solutions of such problems [5] depend also on the dimensionless complex $\tau = Ut/x$. Moreover, as a simple analysis shows, the use of the method for periodic boundary layers is possible only with small Strouhal numbers.

NOTATION

x, y	are the longitudinal and transverse coordinates in the boundary layer;
t	is the time;
η	is the dimensionless transverse coordinate;
$U(x, t)$	is the velocity at outer limit of boundary layer;
ψ	is the stream function;
φ	is the dimensionless stream function;
u, v	are the projections of velocity in boundary layer onto x and y axes, respectively;
ρ	is the density of liquid;
μ, ν	are the coefficients of dynamic and kinematic viscosity;
h	is the scale of transverse coordinate in boundary layer;
$z = h^2/\nu$, H^*, and H^{**}	are the characteristic functions;
δ^*	is the displacement thickness;
δ^{**}	is the thickness of momentum loss;
τ_w	is the stress of surface friction;
ζ	is the reduced coefficient of friction;
B	is the normalizing factor;
f_{kn}, g_{lm}	are the dimensionless parameters;
a, b	are the constants.

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